

## REMARK ON TONO'S THEOREM ABOUT CUSPIDAL CURVES

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*to Vladimir Lin in occasion of his 80-th birthday*

Lin and Zaidenberg [4, 5] (see also [1; §5], [3], and Remark 2 below) asked the following questions.

(Q1) *Does there exist a connection between the topology of an irreducible plane affine algebraic curve and the number of its irreducible singularities?* (Q2) *Is it true, for example, that the number of irreducible singularities of such a curve  $A$  does not exceed  $1 + 2b_1(A)$  where  $b_1(A)$  is the first Betti number of  $A$ ?*

Conjecturally, the answer to the both questions is positive. The first and the most fascinating case of this conjecture was proven by Lin and Zaidenberg themselves [11]: if  $b_1(A) = 0$ , then an automorphism of  $\mathbb{C}^2$  transforms  $A$  into  $x^p = y^q$ , in particular,  $A$  has at most one singular point. Borodzik and Żołądek [2] proved that the answer to Question (Q2) is positive in one more particular case. Namely, if  $A$  is homeomorphic to an annulus, then  $A$  has at most three singular points.

If we pass from  $A$  to its closure in  $\mathbb{CP}^2$ , then the number of singular points may only increase whereas the first Betti number may only decrease. Thus a positive answer to (Q1) follows from the analogous conjecture for plane projective curves. A particular case of the projective conjecture was proven in [12]: if a projectively rigid curve in  $\mathbb{CP}^2$  is homeomorphic to a sphere, then it has at most 9 singular points. Then Tono [8] proved a much stronger result: if a curve in  $\mathbb{CP}^2$  is homeomorphic to a Riemann surface of genus  $g$ , then it has no more than  $(21g + 17)/2$  singular points (thus no more than 8 when  $g = 0$ ).

The purpose of this note is to remark that Tono's arguments extend without changes to the case of an arbitrary plane projective curve and they prove the projective analog of the conjecture and hence, a positive answer to Question (Q1).

Let us give precise statements. Let  $C$  be an algebraic curve in  $\mathbb{CP}^2$ . A singular point of  $C$  is called a *cusp* if  $C$  has a single local analytic branch at it. Let  $s$  be the number of all singular points of  $C$  and  $c$  the number of cusps. Let  $b_i = b_i(C)$  be the  $i$ -th Betti number of  $C$ . So,  $b_2$  is the number of irreducible components. Let  $g = g(C)$  be the total genus of  $C$ , i.e., the sum of the genera of the normalizations of all the irreducible components.

**Theorem 1.** *If  $\bar{\kappa}(\mathbb{CP}^2 \setminus C) = 2$  (by [9] this is so, for example, when one of irreducible components of  $C$  has  $\geq 3$  singular points), then  $c \leq \frac{9}{2}b_1 + \frac{3}{2}g - 6b_2 + \frac{29}{2}$  and  $s \leq \frac{11}{2}b_1 - \frac{1}{2}g - 5b_2 + \frac{27}{2}$ .*

**Corollary 1.** *If  $C$  is irreducible, then  $c \leq \frac{9}{2}b_1 + \frac{3}{2}g + \frac{17}{2} \leq \frac{21}{4}b_1 + \frac{17}{2}$  and  $s \leq \frac{11}{2}b_1 - \frac{1}{2}g + \frac{17}{2} \leq \frac{11}{2}b_1 + \frac{17}{2}$ .*

Let  $C^{\text{aff}}$  be the intersection of  $C$  with some fixed affine chart and let  $b_i^{\text{aff}} = b_i(C^{\text{aff}})$ . We denote the number of singular points, the number of cusps, and the number of points at infinity of  $C^{\text{aff}}$  by  $s^{\text{aff}}$ ,  $c^{\text{aff}}$ , and  $p$  respectively.

**Corollary 2.** *If  $\bar{\kappa}(\mathbb{CP}^2 \setminus C) = 2$ , then  $c^{\text{aff}} \leq \frac{9}{2}(b_1^{\text{aff}} - b_0^{\text{aff}} - p) + \frac{3}{2}(g - b_2) + 19$  and  $s^{\text{aff}} \leq \frac{11}{2}(b_1^{\text{aff}} - b_0^{\text{aff}} - p) + \frac{1}{2}(b_2 - g) + 19$ .*

**Corollary 3.** *If  $C$  is irreducible, then  $c^{\text{aff}} \leq \frac{9}{2}(b_1^{\text{aff}} - p) + \frac{3}{2}g + 13 \leq \frac{9}{2}b_1^{\text{aff}} + \frac{3}{2}g + \frac{17}{2}$  and  $s^{\text{aff}} \leq \frac{11}{2}(b_1^{\text{aff}} - p) - \frac{1}{2}g + 14 \leq \frac{11}{2}b_1^{\text{aff}} - \frac{1}{2}g + \frac{17}{2}$ .*

*Proof of Theorem 1.* Let  $\sigma : V \rightarrow \mathbb{CP}^2$  be the birational morphism (a composition of blowups) such that  $D = \sum D_i = \sigma^{-1}(C)$  is a curve with simple normal crossings all whose irreducible components  $D_i$  are smooth. Let  $K$  be the canonical class of  $V$  and let  $p_a(D) = D(K + D)/2 + 1$ . Then [8; Corollary 4.4] combined with the lower bound  $2c$  for the number of maximal twigs (proved by the same arguments as in [8; p. 220]) yields

$$2c \leq 12e(V \setminus D) + 5 - 3p_a(D) \quad (1)$$

where  $e$  stands for the Euler characteristic. Let  $\Gamma$  be the dual graph of  $D$ , i.e., the vertices of  $\Gamma$  correspond to the irreducible components of  $D$  and the edges correspond to the crossing points. Let  $b_1^\Gamma = b_1(\Gamma)$ . Then we have  $p_a(D) = g + b_1^\Gamma$  and  $b_1 = b_1(D) = 2g + b_1^\Gamma$  whence  $p_a(D) = b_1 - g$ . We have also  $e(V \setminus D) = e(\mathbb{CP}^2 \setminus C) = 2 + b_1 - b_2$ . Thus (1) yields the required bound for  $c$ .

Let  $d = \sum_{i=1}^s (r_i - 1)$  where  $r_i$  is the number of local analytic branches of  $C$  at the  $i$ -th singular point. We have  $d \geq s - c$ . We define the *incidence graph* of a curve as the bipartite graph whose vertices correspond to singular points and to irreducible components of the curve, and the edges correspond to its local branches at singular points: the edge corresponding to a local branch at  $p$  connects the vertex corresponding to  $p$  with the vertex corresponding to the irreducible component containing the local branch. Let  $\Gamma_C$  and  $\Gamma_D$  be the incidence graphs of  $C$  and  $D$ . It is clear that  $\Gamma_D$  is homeomorphic to  $\Gamma$ . Since the topology of incidence graphs does not change under blowups, it follows that  $\Gamma_C \cong \Gamma_D \cong \Gamma$ . Further,  $\Gamma_C$  has  $b_2 + s$  vertices and  $d + s$  edges, thus  $e(\Gamma_C) = b_2 - d$ . Since  $e(\Gamma) = 1 - b_1^\Gamma = 1 - b_1 + 2g$ , we obtain

$$s \leq c + d = c + b_2 - e(\Gamma) = c + b_2 + b_1 - 2g - 1$$

and the result follows from the bound for  $c$ .

*Proof of Corollaries. 1.* Since the upper bounds for  $c$  and  $s$  are  $\geq 3$ , it is enough to consider the case when  $s \geq 3$  and hence  $\bar{\kappa}(\mathbb{CP}^2 \setminus C) \geq 2$  by [9].

**2, 3.** Use  $c^{\text{aff}} \leq c$ ,  $s^{\text{aff}} \leq s$ , and

$$b_1 = b_2 + 1 - e(C) = b_2 + 1 - e(C^{\text{aff}}) - p = b_2 + 1 - b_0^{\text{aff}} + b_1^{\text{aff}} - p$$

in Theorem 1 and Corollary 1.

**Remark 1.** In a recent preprint [7; Theorem 1.4], Palka improved Tono's estimate  $c \leq 8$  in the case of rational cuspidal curves up to  $c \leq 6$ . Maybe, his arguments could give a better upper bound for  $c$  and  $c^{\text{aff}}$  in the general case.

**Remark 2.** In the case of an irreducible affine curve, Corollary 3 gives an estimate  $c^{\text{aff}} \leq \alpha b_1^{\text{aff}} + \beta$  with  $(\alpha, \beta) = (5.25, 8.5)$ . This estimate does not seem to be optimal. The example  $y^2 = p(x)^3$  shows that one cannot do better than  $(\alpha, \beta) = (1, 1)$  (see also [1]). Note that the projective dual of a smooth or nodal cubic with a suitable choice of the infinite line are the only known exceptions for the estimate with  $(\alpha, \beta) = (1, 1)$ ; for them we have  $b_1^{\text{aff}} = 1, 5, 6$ , or  $7$ .

**Remark 3.** Let  $D = D_1 + \cdots + D_n$  be a reduced curve with simple normal crossings on a smooth algebraic surface  $V$  and let  $\Gamma$  be the dual graph of  $D$ . We set  $\beta(D_i) = D_i(D - D_i)$  (the degree of the corresponding vertex of  $\Gamma$ ). If  $\beta(D_i) = 1$ , we say that  $D_i$  is a *tip* of  $D$ . We assume that  $D$  does not contain any rational  $(-1)$ -curve  $D_i$  with  $\beta(D_i) \leq 2$ . Zaidenberg [10; p. 16] conjectured that *only finite number of pairwise non-homeomorphic graphs  $\Gamma$  can be obtained in this way under the condition that  $\bar{\kappa}(V \setminus D) = 2$  and  $b_i(V \setminus D) = 0$  for  $i > 0$ .*

Similarly to Theorem 1, this conjecture follows immediately from [8; Corollary 4.4]. Indeed, the condition  $b_i(V \setminus D) = 0$ ,  $i > 0$ , implies that  $\Gamma$  is a tree. Hence it is enough to bound  $t_D$  (the number of tips of  $D$ ). A bound  $t_D \leq 17$  follows from [8; Corollary 4.4] provided that the pair  $(V, D)$  is *almost minimal* which in our setting is equivalent to the absence of rational  $(-1)$ -curves  $E$  such that  $E \not\subset D$  and  $ED \leq 1$  (see [8; Lemma 3.4]). If such  $E$  exists, let  $\pi : V \rightarrow V'$  be its blowing down and let  $D' = \pi_*(D)$ . Then  $\bar{\kappa}(V' \setminus D') = 2$  by [8; Lemma 3.2] and  $e(V' \setminus D') \leq e(V \setminus D) - 1 = 0$ . This contradicts [6; Theorem 6.7.1].

There are 3901520 trees with at most 17 tips (the one vertex tree included).

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